1. When we write $x = \frac{m}{n}$ for a positive rational we assume implicitly that $m, n \in \mathbb{N}$ and they do not have common divisors (x is represented "canonically" or "normally represented"). Fixing any finite open interval contained I in $(0, \infty)$ (say of length 1, let

$$B_n = \{x = \tfrac{m}{n} \in \mathbb{Q} \cap (0,\infty)\} \cap I, \quad \forall n \in \mathbb{N}$$

Show that B_n is a finite set ($\forall n \in \mathbb{N}$) and, in fact, $\#(B_n) \leq n$

2. Let $f: (0,\infty) \to \mathbb{R}$ be defined by $f(x) = 0$ if $x \notin \mathbb{Q}$ and $f(x) = n$ if $x = \frac{m}{n}$ as in Q1. Show that f is not bounded on any interval of pos. length.

Q4. Let $f : \mathbb{R} \to \mathbb{R}$ be additive: $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. Suppose f is continuous at $x_0 = 0$. Show that f is continuous everywhere and $f(x) = cx \ \forall x$ where $c = f(1)$.

Q5 Let $f(r) = 0$ for all $r \in \mathbb{Q}$. Suppose f is continuous on $\mathbb{R}$. Show that $f(x) = 0$ for all $x \in \mathbb{R}$.

Let $g : \mathbb{R} \to \mathbb{R}$ be such that

$$g(x) = \begin{cases} 2x & \text{if } x \in \mathbb{Q}; \\ x+3 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Find the continuity points of g.

3. Let $f : (0,\infty) \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 0 & \text{if } x \notin \mathbb{Q}; \\ 1/q & \text{if } x = \frac{p}{q} \text{ (in the representation of } x). \end{cases}$$

Show that f is continuous at x_0 if and only if x_0 is irrational.

Q 6 Let $f : A \to \mathbb{R}$, $x_0 \in \mathbb{R}$ non-isolated to A and suppose that $\lim_{x \to x_0} f(x)$ does not exist. Show that there exist $\varepsilon > 0$ and two sequences (x_n), (y_n) in $A \setminus \{x_0\}$ converge to x_0 such that $|f(x_n) - f(y_n)| \geq \varepsilon$ for all n.

If f is bounded (in the sense that the range of f is a bounded subset of $\mathbb{R}$), show further that there exist two sequences (x'_n) and (y'_n) in $A \setminus \{x_0\}$ converge to x_0 such that $\lim f(x'_n) = \ell' \neq \ell'' = \lim f(y'_n)$.

Q7 Consider real numbers $a < b < c$. Let $f : (a, b] \to \mathbb{R}$ and $g : [b, c) \to \mathbb{R}$ be continuous at b, and suppose that $f(b) = g(b)$. Let $h : (a, c) \to \mathbb{R}$ be defined by

$$h(x) = \begin{cases} f(x) & \text{if } x \in (a, b]; \\ g(x) & \text{if } x \in [b, c). \end{cases}$$

Show that

(a) h is continuous at b;

(b) if f, g are uniformly continuous then h is uniformly continuous.

Q8 (is not required for you to do but is my X'mas present for some). In light of Q3 (Thomae function), it is nature to ask; whether or not to have a function which is continuous at and only at rationals? (the answer is "No": no such function!)

Q9. (is not required). Does the addivity in Q4 imply the continuity?